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Quasistatic delamination problem

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Abstract We study delamination of two elastic bodies glued together by an adhesive that can undergo a unidirectional inelastic rate-independent process. The quasistatic delamination process is thus activated by time-dependent external loadings, realized through body forces and displacements prescribed on parts of the boundary. The novelty of this work consists of considering the glue as infinitesimally thin and ideally rigid in the sense that a crack in the glue cannot be seen before, speaking "microscopically", all macromolecular links of the adhesive are fully debonded. The concept of energetic solution is applied and existence of such solutions is proved by showing Γ-convergence of a suitable approximation that, in addition, allows for a direct computer implementation, unlike the original problem.

Keywords: delamination, debonding, rate-independent processes, energetic formulation, Γ-convergence.

Mathematics Subject Classification (2000): 49J40, 74G65, 74R05, 74M15, 35K90

1 Introduction

We are interested in the problem of several elastic bodies glued together by an adhesive that can undergo an inelastic process of "delamination" (sometimes also called "debonding"). Upon loading, "microscopically" speaking, some macromolecules in the adhesive may break and we assume that they can never be glued back, i.e., no "healing" is possible. This makes the process *unidirectional* (sometimes also called irreversible) and its analysis becomes more difficult than if healing is allowed. On the glued surface, we consider the *delamination process as rate-independent* and, in the bulk, we neglect all inertial or viscous-like effects, so that the whole problem is rate-independent. Moreover, we confine ourselves to *small strains* and, just for a notational simplicity, we consider only two bodies Ω_1 and Ω_2 glued together along the *contact surface* $\Gamma_{\rm c}$. At a current time, the "volume fraction" of debonded molecular links will be "macroscopically" described by the *scalar delamination parameter* $z : \Gamma_{\rm c} \to [0, 1]$. The state $z(x) = 0$ means that the surface is completely

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debonded at $x \in \Gamma_{\rm c}$, the intermediate state $0 < z(x) < 1$ means that there are some molecular links which have been broken but the remaining ones are effective, while $z(x) = 1$ means that the adhesive is still 100% undestroyed and thus fully effective. Activating the delamination process at a given point $x \in \Gamma_{\rm c}$ needs a certain (phenomenologically prescribed) energy per unit area $a(x) = a_0(x) + a_1(x)$ whose part a_0 is considered as stored into increase of energy by debonding the surface at x and another part a_1 as dissipated, e.g. into chaotic lattice vibrations (i.e., heat, cf. Remark 4.2 below).

We consider here the case of linearized elasticity for some energy functional $\mathcal{E}_\infty$ and of a *non-symmetric* dissipation potential $\mathcal{R}$, see (2.10) and (2.7) below. Moreover, we take into account the local non-interpenetration of matter by asking, for the displacement u, that $[u]_{\Gamma_{\rm c}} \cdot \nu \geq 0$ on $\Gamma_{\rm c}$, where ν denotes the unit normal to $\Gamma_{\rm c}$ oriented, say, from Ω_2 to Ω_1 and then $[u]_{\Gamma_{\rm c}} = u^+ - u^-$, with u^+ (resp. u^-) being the trace on $\Gamma_{\rm c}$ of the restriction of u to Ω_1 (resp. to Ω_2).

The aim of this work is to obtain existence of special weak, so-called energetic, solutions (u, z) associated with $\mathcal{E}_\infty$ and $\mathcal{R}$ such that

$$[u(x)]_{\Gamma_{\rm c}} = 0 \text{ for a.e. } x \text{ such that } z(x) > 0. \tag{1.1}$$

This condition models the glue as infinitesimally thin and ideally rigid in the sense that a crack in the glue at the "macroscopic" point $x \in \Gamma_{\rm c}$ cannot be seen before all macromolecular links of the adhesive are fully debonded, i.e., before $z(x)$ attains 0 at some time. Thus, at a current time, the crack set is defined by $\{\, x \in \Gamma_{\rm c} \,|\, z(x) = 0 \,\}$. For the concept of energetic solutions we refer to [20,23–25].

Existence of a solution as well as its direct numerical implementation does not seem straightforward because of the discontinuous character of the condition (1.1) related with the ideal thinness and rigidity of the glue. Therefore the idea is to consider first energetic solutions (u_k, z_k) associated with a modified energy functional $\mathcal{E}_k$ and with the same dissipation potential $\mathcal{R}$, and then, after proving that $\mathcal{E}_k$ Γ-converges to $\mathcal{E}_\infty$, apply the results in [22] obtaining an energetic solution (u, z) associated with the original $\mathcal{E}_\infty$ and $\mathcal{R}$. The modified energy functional $\mathcal{E}_k$ we will consider (see (2.14) below) actually differs from $\mathcal{E}_\infty$ via the surface term

$$\int_{\Gamma_{\rm c}} kz[u]^2_{\Gamma_{\rm c}} \,\mathrm{d}\mathscr{H}^{d-1}, \qquad \text{where we abbreviate } \ [u]^2_{\Gamma_{\rm c}} := \big|[u]_{\Gamma_{\rm c}}\big|^2, \tag{1.2}$$

replacing the theoretically and numerically troublesome condition (1.1) and representing the elastic energy stored in the adhesive, as also previously considered, e.g., in [18,19]. In particular, existence of an energetic solution (u_k, z_k) for $\mathcal{E}_k$ and $\mathcal{R}$ has been already established.

Notice that the energy $\mathcal{E}_k$ can be finite even for a displacement u for which $z[u]_{\Gamma_{\rm c}} \neq 0$ on a subset of $\Gamma_{\rm c}$ with positive $\mathscr{H}^{d-1}$-measure. However, the product $z[u]^2_{\Gamma_{\rm c}}$ in (1.2) is highly penalized by the factor k, so that the constraint (1.1) is obtained in the limit when $k \to \infty$.

Our model is related to [16,17,26,27], where the evolution of a single crack with prescribed path is studied, in the case of a single crack tip. The present model allows for more general crack sets although, for some specific loadings, it seems possible that it would give the same response as the models mentioned above. The approximating (and regularized) problem containing the surface term (1.2) has been considered already in [18,19], and is also related to [4,5,9], where a prescribed crack path is considered for cohesive-zone models describing delamination with partially debonded crack surfaces. Moreover, according to Barenblatt's cohesive-zone model [1], the energy density needed to produce a new crack (or to increase an existing one) depends on the crack opening, namely on $[u]_{\Gamma_{\rm c}}$.

Following the approach in [3], one might consider a non-negative measure μ taking values in $\{0,\infty\}$, instead of the delamination parameter z. In this case, however, it is not clear how to define the dissipation distance. Moreover, no implementable numerical scheme has been devised for such model.

2 The model, its energetic formulation and approximation

Let $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ be a Lipschitz domain, and let us consider its decomposition $\Omega = \Omega_1 \cup \Gamma_{\rm c} \cup \Omega_2$, where Ω_1 and Ω_2 are two disjoint Lipschitz sub-domains and $\Gamma_{\rm c}$ is their common boundary. Thus, $\Gamma_{\rm c}$ represents a prescribed delamination $(d{-}1)$-dimensional surface. We assume that $\partial\Omega$ is the union of two disjoint subsets $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$, with

$$\mathscr{H}^{d-1}(\partial\Omega_i \cap \Gamma_{\rm D}) > 0, \qquad i = 1,2, \tag{2.1}$$

where $\mathscr{H}^{d-1}$ denotes the $(d{-}1)$-dimensional Hausdorff measure. Moreover, we assume $\overline{\Gamma}_{\rm c} \cap \overline{\Gamma}_{\rm D} = \emptyset$, cf. Figure 1. On the Dirichlet part of the boundary $\Gamma_{\rm D}$ we impose a time-dependent boundary displacement $w_{\rm D}(t)$, while the remaining part $\Gamma_{\rm N}$ is assumed to be free. Therefore, any admissible displacement $\widetilde{u} : \Omega \setminus \Gamma_{\rm c} \to \mathbb{R}^d$ has to be equal to a prescribed "hard-device" loading $w_{\rm D}(t)$ on $\Gamma_{\rm D}$.

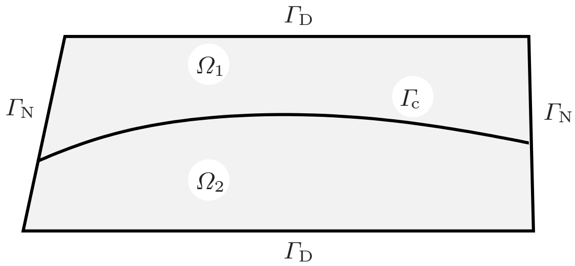

Fig. 1. Illustration of the geometry and of the notation.

As already outlined in Section 1, we assume, like in [18,19], that Ω_1 and Ω_2 are glued together along the surface $\Gamma_{\rm c}$, and that the volume fraction of the debonded molecular links in the adhesive is represented by the scalar delamination parameter $z : \Gamma_{\rm c} \to [0,1]$.

We consider the case of linearized elasticity and assume that the elastic energy $\mathcal{V}_0$ stored in the volume Ω is

$$\mathcal{V}_0(\widetilde{u}) := \frac{1}{2}\int_{\Omega_1\cup\Omega_2} \mathbb{C}e(\widetilde{u}){:}e(\widetilde{u})\,\mathrm{d}x = \frac{1}{2}\sum_{i=1,2}\int_{\Omega_i} \mathbb{C}e(\widetilde{u}){:}e(\widetilde{u})\,\mathrm{d}x, \tag{2.2}$$

where $\mathbb{C} \in \mathrm{Lin}(\mathbb{M}^{d\times d}_{\rm sym})$ represents the elasticity tensor, and $\mathbb{M}^{d\times d}_{\rm sym}$ is the set of $(d{\times}d)$-symmetric matrices, while $e(\widetilde{u}) := \frac{1}{2}(\nabla\widetilde{u})^\top + \frac{1}{2}\nabla\widetilde{u}$ is the small-strain tensor. We assume that $\mathbb{C}$ is a positive definite tensor, i.e.,

$$\exists\,\mu > 0 \;\; \forall A \in \mathbb{M}^{d\times d}_{\rm sym} : \qquad \mathbb{C}A{:}A \geq \mu|A|^2. \tag{2.3}$$

In order to make possible the application of [22, Theorem 3.1], we assume the prescribed boundary displacement $w_{\rm D}$ and the applied bulk force f to be qualified as

$$w_{\rm D}{\in}C^1([0,T];H^{1/2}(\Gamma_{\rm D};\mathbb{R}^d)), \quad f{\in}C^1([0,T];L^r(\Omega;\mathbb{R}^d)), \quad r > \begin{cases} 6/5 & \text{if } d=3, \\ 1 & \text{if } d=2. \end{cases} \tag{2.4}$$

As already introduced in Sect. 1, $[u]_{\Gamma_{\rm c}} = u^+ {-} u^-$ is the jump of u across $\Gamma_{\rm c}$, and $[u]_{{\rm n},\Gamma_{\rm c}} = [u]_{\Gamma_{\rm c}}{\cdot}\nu$ is the jump of the normal displacement across $\Gamma_{\rm c}$. We recall that ν

denotes the unit normal to $\Gamma_{\rm c}$ oriented from Ω_2 to Ω_1 and that u^+ (resp. u^-) is the trace on $\Gamma_{\rm c}$ of the restriction of u to Ω_1 (resp. to Ω_2).

The state of the system is represented by an element $q = (\widetilde{u}, z)$ of the state space $\mathcal{Q}$, defined by

$$\mathcal{Q} := \mathcal{U} \times \mathcal{Z}, \qquad \text{with} \qquad \mathcal{U} := H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d), \qquad \mathcal{Z} := L^\infty(\Gamma_{\rm c}). \tag{2.5}$$

We define the stored energy functional $\widetilde{\mathcal{E}}_\infty : [0,T] \times \mathcal{Q} \to \mathbb{R} \cup \{\infty\}$ as

$$\widetilde{\mathcal{E}}_\infty(t,\widetilde{u},z) := \begin{cases} \widetilde{\mathcal{V}}(t,\widetilde{u}) - \displaystyle\int_{\Gamma_{\rm c}} a_0 z \,\mathrm{d}\mathscr{H}^{d-1} & \text{if } \widetilde{u} = w_{\rm D}(t) \text{ on } \Gamma_{\rm D}, \\ & [\widetilde{u}]_{{\rm n},\Gamma_{\rm c}} \ge 0 \text{ and } 0 \le z \le 1 \text{ on } \Gamma_{\rm c}, \\ & [\widetilde{u}(x)]_{\Gamma_{\rm c}} = 0 \text{ if } z(x) > 0, \\ \infty & \text{elsewhere}, \end{cases} \tag{2.6a}$$

$$\text{with } \widetilde{\mathcal{V}}(t,\widetilde{u}) := \mathcal{V}_0(\widetilde{u}) - \int_{\Omega_1\cup\Omega_2} f(t)\cdot\widetilde{u}\,\mathrm{d}x = \int_{\Omega_1\cup\Omega_2} \frac{1}{2}\,\mathbb{C}e(\widetilde{u}):e(\widetilde{u}) - f(t)\cdot\widetilde{u}\,\mathrm{d}x, \tag{2.6b}$$

where $a_0(x) \ge 0$ is the energy per unit $(d{-}1)$-dimensional area deposited "microscopically" into broken interatomic links of the delaminated adhesive. Along the glue, we assume that the energy needed for switching $z(x)$ from 1 to 0 is irreversibly dissipated. Namely, we consider the case when delamination is a unidirectional process allowing no healing, and put

$$\mathcal{R}(\dot{z}) := \int_{\Gamma_{\rm c}} \delta_{(-\infty,0]}(\dot{z}) - a_1 \dot{z} \,\mathrm{d}\mathscr{H}^{d-1} = \begin{cases} -\int_{\Gamma_{\rm c}} a_1 \dot{z}\,\mathrm{d}\mathscr{H}^{d-1} & \text{if } \dot{z} \le 0 \text{ a.e. on } \Gamma_{\rm c}, \\ \infty & \text{elsewhere}, \end{cases} \tag{2.7}$$

where $a_1(x) \ge a_{1,\min} > 0$ is an energy per unit $(d{-}1)$-dimensional area dissipated by delaminating the surface at $x \in \Gamma_{\rm c}$. Once delaminated, the surface cannot be glued back, and this irreversibility behavior ultimately causes the non-symmetry of $\mathcal{R}$.

Here it is convenient, like in [13, Sect.4] or [16], to use the additive split $\widetilde{u} = u + u_{\rm D}(t)$, $u_{\rm D}(t)$ being an extension of $w_{\rm D}(t)$ to Ω. In view of (2.4) and of the assumption $\overline{\Gamma}_{\rm c} \cap \overline{\Gamma}_{\rm D} = \emptyset$, we can assume

$$u_{\rm D} \in C^1([0,T]; H^1(\Omega;\mathbb{R}^d)) \quad \& \quad u_{\rm D}|_{\Gamma_{\rm c}} = 0. \tag{2.8}$$

Note that, in particular, $[u + u_{\rm D}(t)]_{\Gamma_{\rm c}} = [u]_{\Gamma_{\rm c}}$. This will allow for the construction of a joint recovery sequence (see (3.17)) and for passing to the limit in the regularized problems. We remark that the regularized problems admit solutions even if $\overline{\Gamma}_{\rm c} \cap \overline{\Gamma}_{\rm D} \neq \emptyset$, cf. [18].

Accordingly, we thus define the energy functional $\mathcal{E}_\infty : [0,T] \times \mathcal{Q} \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{E}_\infty(t,u,z) := \widetilde{\mathcal{E}}_\infty(t, u + u_{\rm D}(t), z). \tag{2.9}$$

Note that, when $\mathcal{E}_\infty(t,u,z) < \infty$, the displacement u has to be equal to zero on $\Gamma_{\rm D}$, in the sense of traces. In view of (2.6), the transformed energy $\mathcal{E}_\infty$ from (2.9) takes the form

$$\mathcal{E}_\infty(t,u,z) = \begin{cases} \mathcal{V}(t,u) - \displaystyle\int_{\Gamma_{\rm c}} a_0 z \,\mathrm{d}\mathscr{H}^{d-1} & \text{if } u = 0 \text{ on } \Gamma_{\rm D}, \\ & [u]_{{\rm n},\Gamma_{\rm c}} \ge 0 \text{ and } 0 \le z \le 1 \text{ on } \Gamma_{\rm c}, \\ & [u(x)]_{\Gamma_{\rm c}} = 0 \text{ if } z(x) > 0, \\ \infty & \text{elsewhere}, \end{cases} \tag{2.10a}$$

$$\text{with } \mathcal{V}(t,u) := \int_{\Omega_1\cup\Omega_2} \frac{1}{2}\,\mathbb{C}e(u{+}u_{\rm D}(t)):e(u{+}u_{\rm D}(t)) - f(t)\cdot(u{+}u_{\rm D}(t))\,\mathrm{d}x. \tag{2.10b}$$

We are interested in the notion of energetic solution associated with $\mathcal{E}_\infty$ and $\mathcal{R}$ and with an initial condition $q_0 = (u_0, z_0) \in \mathcal{Q}$, as introduced by Mielke et al. [23–25] (see also the survey [20]). We say that the process $q(t) = (u(t), z(t))$ is *stable* for $(\mathcal{E}_\infty, \mathcal{R})$ at time $t \in [0, T]$, if

$$\mathcal{E}_\infty(t, u(t), z(t)) \le \mathcal{E}_\infty(t, \hat{u}, \hat{z}) + \mathcal{R}(\hat{z} - z(t)) \quad \forall\, (\hat{u}, \hat{z}) \in \mathcal{Q}. \tag{2.11}$$

Definition 2.1 ***(Energetic solution.)*** *We call $q = (u, z) : [0, T] \to \mathcal{Q}$ an energetic solution of the initial-value problem determined by the quadruple $(\mathcal{E}_\infty, \mathcal{R}, u_0, z_0)$ if*
(i) $q(0) = (u(0), z(0)) = (u_0, z_0)$,
(ii) $t \mapsto \partial_t \mathcal{E}_\infty(t, q(t)) \in L^1(0, T)$,
(iii) *$q(t)$ is stable for $(\mathcal{E}_\infty, \mathcal{R})$ at time t in the sense of* (2.11),
(iv) $\mathcal{E}_\infty(t, q(t)) + \mathrm{Var}_\mathcal{R}(q, [0, t]) = \mathcal{E}_\infty(0, q(0)) + \int_0^t \partial_s \mathcal{E}_\infty(s, q(s))\,\mathrm{d}s$ *for all* $t \in [0, T]$.
The $\mathcal{R}$-variation $\mathrm{Var}_\mathcal{R}(q, [0, t])$ of the function $q(\cdot) = (u(\cdot), z(\cdot))$ over the time interval $[0, t]$ is defined as

$$\mathrm{Var}_\mathcal{R}(q, [0, t]) := \sup \sum_{i=1}^{N} \mathcal{R}\big(z(t_i) - z(t_{i-1})\big)$$

where the supremum is taken over all $N \in \mathbb{N}$ and over all partitions $0 = t_0 < t_1 < \cdots < t_N = t$ of the interval $[0, t]$.

Note that if $\mathrm{Var}_\mathcal{R}(q, [0, t]) < \infty$ then, for $q = (u, z)$, the map $z(\cdot, x)$ is non-increasing in time for a.e. $x \in \Gamma_\mathrm{c}$ and

$$\mathrm{Var}_\mathcal{R}(q, [0, t]) = \mathcal{R}\big(z(t) - z(0)\big). \tag{2.12}$$

Indeed this follows by the definition of $\mathcal{R}$ given in (2.7).

We observe that due to definition (2.10), we have that $\partial_t \mathcal{E}_\infty(t, u, z) = \partial_t \mathcal{V}(t, u)$ when $\mathcal{E}_\infty(t, u, z) < \infty$, where

$$\partial_t \mathcal{V}(t, u) = \int_{\Omega_1 \cup \Omega_2} \mathbb{C}e\big(u + u_\mathrm{D}(t)\big){:}e\big(\partial_t u_\mathrm{D}(t)\big) - \partial_t f(t){\cdot}(u + u_\mathrm{D}(t)) - f(t){\cdot}\partial_t u_\mathrm{D}(t)\,\mathrm{d}x. \tag{2.13}$$

We define the *approximating stored energy functionals* $\mathcal{E}_k : [0, T] \times \mathcal{Q} \to \mathbb{R} \cup \{\infty\}$ as

$$\mathcal{E}_k(t, u, z) := \begin{cases} \mathcal{V}(t, u) + \displaystyle\int_{\Gamma_\mathrm{c}} (k[u]_{\Gamma_\mathrm{c}}^2 - a_0) z \,\mathrm{d}\mathscr{H}^{d-1} & \text{if } \; u = 0 \text{ on } \Gamma_\mathrm{D}, \\ & [u]_{\mathrm{n}, \Gamma_\mathrm{c}} \ge 0 \text{ and } 0 \le z \le 1 \text{ on } \Gamma_\mathrm{c}, \\ \infty & \text{elsewhere}, \end{cases} \tag{2.14}$$

with $\mathcal{V}$ defined in (2.10b). We notice that the regularizing term in $\mathcal{E}_k$, describing the elastic energy stored in the adhesive, is the same as in (1.2). Moreover, the *regularized problem* can actually be implemented on computers after additional time-spatial discretization, as documented in [18, Proposition 3.3].

Energetic solutions to $(\mathcal{E}_k, \mathcal{R}, q_0)$ are defined analogously as before, i.e., by Definition 2.1 but with $\mathcal{E}_\infty$ replaced by $\mathcal{E}_k$. We note that $\partial_t \mathcal{E}_k = \partial_t \mathcal{V} = \partial_t \mathcal{E}_\infty$.

Proposition 2.2 ***(Existence of energetic solutions to regularized problems [18].)*** *Let (2.1), (2.3), (2.4), and (2.8) be satisfied and let $q_0^k = (u_0^k, z_0^k)$ be a sequence of initial conditions, with q_0^k stable for $(\mathcal{E}_k, \mathcal{R})$ at time $t = 0$. Then for every $k \in \mathbb{N}$ there exists an energetic solution (u_k, z_k) associated with $(\mathcal{E}_k, \mathcal{R}, u_0^k, z_0^k)$.*

This existence result has essentially been proved in [18, Proposition 3.3]. The proof consists of approximating the continuous-time problem by discrete-time problems, for which the solution is defined via an incremental minimization method. Indeed, for each $k \in \mathbb{N}$, let us fix a sequence of partitions $\{0 = t_0^n < t_1^n < \cdots < t_n^n = T\}$ of the interval $[0, T]$ with $\max_{1 \le i \le n}(t_i^n - t_{i-1}^n) \to 0$ as $n \to \infty$. For every $n \in \mathbb{N}$ the pair (u_i^n, z_i^n) is defined by induction as follows. For $i = 0$ we set $(u_0^n, z_0^n) := (u_0^k, z_0^k)$, while for $i = 1, \ldots, n$ we define

$$(u_i^n, z_i^n) \in \text{ Argmin } \{ \mathcal{E}_k(t_i^n, u, z) + \mathcal{R}(z - z_{i-1}^n) \mid (u, z) \in \mathcal{U} \times \mathcal{Z} \}.$$

Existence of the "time-discrete" solution (u_i^n, z_i^n) follows from lower semicontinuity properties and coercivity. Then it is convenient to define the piecewise constant interpolants associated with the discrete solution via $(u^n(t), z^n(t)) := (u_i^n, z_i^n)$ for $t \in [t_i^n, t_{i+1}^n)$. A suitable choice for a test function provides the *a priori* estimate $\|u^n\|_{L^\infty(0,T;\mathcal{U})} \le C$, while the *a priori* estimate $\|z^n\|_{L^\infty(0,T;\mathcal{Z})} \le C$ follows easily from the fact that $|z^n(t)| \le 1$ for every n and t. Then there exists a subsequence converging to some limit pair (u, z) and one has to show that it is an energetic solution associated with $\mathcal{E}_k$ and $\mathcal{R}$. This is nontrivial but well developed in the literature (see, e.g. [8,13,20] and references therein).

3 *Γ*-convergence of the regularized problems

First, let us recall the notion of Γ-convergence introduced by De Giorgi in [10]. For a comprehensive introduction to this topic, see also [2,7].

In its more general setting, Γ-convergence applies to a family of functionals defined on a topological space, with respect to the convergence induced by the topology. In the particular case of functionals defined on a metric space, the definition of this variational convergence simplifies, by means of the sequential characterization of the convergence.

More precisely, for given $\mathcal{G}, \mathcal{G}_k : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$, where $\mathcal{X}$ is a metric space, we say that the sequence $\{\mathcal{G}_k\}_{k \in \mathbb{N}}$ Γ-converges to $\mathcal{G}$ and that the functional $\mathcal{G}$ is the Γ-limit of $\{\mathcal{G}_k\}_{k \in \mathbb{N}}$ if for every $w \in \mathcal{X}$ the following two conditions are satisfied:

$$\text{(liminf inequality)} \qquad \forall (w_k)_{k \in \mathbb{N}},\ w_k \overset{\mathcal{X}}{\to} w : \mathcal{G}(w) \le \liminf_{k \to \infty} \mathcal{G}_k(w_k); \tag{3.1a}$$

$$\text{(recovery sequence)} \qquad \exists\, (\widehat{w}_k)_{k \in \mathbb{N}},\ \widehat{w}_k \overset{\mathcal{X}}{\to} w : \ \mathcal{G}(w) \ge \limsup_{k \to \infty} \mathcal{G}_k(\widehat{w}_k). \tag{3.1b}$$

In the following we consider the case $\mathcal{G}_k = \mathcal{E}_k(t, \cdot)$, for fixed $t \in [0, T]$ and $\mathcal{X} = \mathcal{Q}$, so that $w = q$. We will derive the Γ- limit of the sequence $(\mathcal{E}_k(t, \cdot))$ with respect to the topology in $\mathcal{Q}$ defined by the convergence

$$q_k \overset{\mathcal{Q}}{\to} q \quad \overset{\text{def}}{\Longleftrightarrow} \quad \begin{cases} u_k \rightharpoonup u & \text{in } \mathcal{U} = H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d), \\ z_k \overset{*}{\rightharpoonup} z & \text{in } \mathcal{Z} = L^\infty(\Gamma_{\mathrm{c}}). \end{cases}$$

Notice that due to the coercivity of the functionals $\mathcal{E}_k(t, \cdot)$, we work essentially with bounded subsets of $\mathcal{Q}$, where the $\mathcal{Q}$-topology is metrizable (since H^1 has a separable dual and L^∞ is the dual of the separable space L^1). Therefore we can equivalently work with sequences.

The following result reveals the functional $\mathcal{E}_\infty$ defined in (2.10) as the correct limit of the functionals $\mathcal{E}_k$ defined in (2.14) or, vice versa, the sequence $(\mathcal{E}_k)$ as the correct approximation of $\mathcal{E}_\infty$.

Lemma 3.1 *(**Γ-limit of the stored energy.**) Let $t \in [0, T]$. Under assumptions* (2.1), (2.3), (2.4), *and* (2.8), *the sequence $\{\mathcal{E}_k(t, \cdot)\}_{k \in \mathbb{N}}$ Γ-converges (with respect to the $\mathcal{Q}$-topology) to $\mathcal{E}_\infty(t, \cdot)$, where $\mathcal{E}_\infty$ is defined in* (2.10).

Proof Compactness of sequences with equibounded energy. Let $q_k = (u_k, z_k) \subset \mathcal{Q}$ be a sequence with equibounded energy, i.e.,

$$\sup_k \mathcal{E}_k(t, q_k) \le c < \infty. \tag{3.2}$$

Then we can deduce in particular that, for $i = 1, 2$, and for every k,

$$\frac{1}{2} \int_{\Omega_i} \mathbb{C}e(u_k + u_D(t)) : e(u_k + u_D(t))\, \mathrm{d}x \le c. \tag{3.3}$$

Therefore, since $u_k = 0$ on $\partial\Omega_i \cap \Gamma_D$, and $\mathscr{H}^{d-1}\big(\partial\Omega_i \cap \Gamma_D\big) > 0$ for $i = 1, 2$ by (2.1), assumptions (2.3), (2.4), (2.8), and Korn's inequality (see [15]) imply

$$\sup_k \|u_k\|_{H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d)} \le c. \tag{3.4}$$

Hence, there exists $u \in \mathcal{U} = H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ such that

$$u_k \rightharpoonup u \quad \text{in } \mathcal{U}.$$

This entails in particular the convergence of the traces of u_k on Γ_c to the corresponding traces of the limit function u, i.e.,

$$u_k^+ \to u^+ \ \text{ and } \ u_k^- \to u^- \quad \text{in } L^2(\Gamma_\mathrm{c}; \mathbb{R}^d). \tag{3.5}$$

Thus also $[u_k]_{\Gamma_\mathrm{c}} \to [u]_{\Gamma_\mathrm{c}}$ in $L^2(\Gamma_\mathrm{c}; \mathbb{R}^d)$. On the other hand, since $\sup_k \|z_k\|_{L^\infty(\Gamma_\mathrm{c})} \le 1$, we directly deduce the existence of $z \in \mathcal{Z} = L^\infty(\Gamma_\mathrm{c})$, $0 \le z \le 1$ a.e. on Γ_c, such that

$$z_k \overset{*}{\rightharpoonup} z \quad \text{in } \mathcal{Z}. \tag{3.6}$$

From (3.5) and (3.6) it follows that

$$\lim_{k\to\infty} \int_{\Gamma_\mathrm{c}} z_k [u_k]_{\Gamma_\mathrm{c}}^2 \, \mathrm{d}\mathscr{H}^{d-1} = \int_{\Gamma_\mathrm{c}} z [u]_{\Gamma_\mathrm{c}}^2 \, \mathrm{d}\mathscr{H}^{d-1}. \tag{3.7}$$

Therefore, since from (3.2) we have in particular that

$$\int_{\Gamma_\mathrm{c}} z_k [u_k]_{\Gamma_\mathrm{c}}^2 \, \mathrm{d}\mathscr{H}^{d-1} \le \frac{c}{k},$$

then, by (3.7) we get

$$\int_{\Gamma_\mathrm{c}} z [u]_{\Gamma_\mathrm{c}}^2 \, \mathrm{d}\mathscr{H}^{d-1} = 0.$$

Hence, in particular, $z[u]_{\Gamma_\mathrm{c}} = 0$ $\mathscr{H}^{d-1}$-a.e. on Γ_c, or, equivalently, since $z \ge 0$, we have that (1.1) is satisfied.

Liminf inequality. Let $q = (u, z) \in \mathcal{Q}$ and let $(q_k)_{k\in\mathbb{N}} = (u_k, z_k)_{k\in\mathbb{N}}$ be such that $(u_k, z_k) \overset{\mathcal{Q}}{\to} (u, z)$ and $\sup_k \mathcal{E}_k(t, u_k, z_k) < \infty$. Then, by the compactness result provided in the first part of the proof we have that (u, z) belongs to the domain where $\mathcal{E}_\infty$ is finite. Moreover, by lower semicontinuity with respect to the convergence in $\mathcal{Q}$,

$$\liminf_{k\to\infty} \mathcal{E}_k(t, u_k, z_k) \ge \mathcal{E}_\infty(t, u, z). \tag{3.8}$$

Limsup inequality. Let $q = (u, z) \in \mathcal{Q}$ be such that $\mathcal{E}_\infty(t, q) < \infty$. The existence of the recovery sequence in (3.1b) (where, clearly, $\mathcal{X} = \mathcal{Q}$, $\mathcal{G}_k = \mathcal{E}_k(t, \cdot)$, and $\mathcal{G} = \mathcal{E}_\infty(t, \cdot)$) is trivial; we can just take $q_k = q$ for every k.

Now we extend the result of the previous theorem, taking into account also the dependence on the parameter $t \in [0,T]$. More precisely, in what follows, $\mathcal{X} = [0,T] \times \mathcal{Q}$, $\mathcal{G}_k = \mathcal{E}_k$, and $w = (t,q)$.

Corollary 3.2 *Under assumptions* (2.1), (2.3), (2.4), *and* (2.8)*, the sequence of functionals* $\{\mathcal{E}_k\}_{k\in\mathbb{N}}$ *Γ-converges (with respect to the* $([0,T]\times\mathcal{Q})$*-topology) to* $\mathcal{E}_\infty$*.*

Proof Limsup inequality. Let $(t,q) \in [0,T] \times \mathcal{Q}$ be such that $\mathcal{E}_\infty(t,q) < \infty$. It is then sufficient to define the sequence $(t_k, q_k) := (t,q)$.

Liminf inequality. Let $t_k \to t$, and $(u_k, z_k) \overset{\mathcal{Q}}{\to} (u,z)$ be such that $\sup_k \mathcal{E}_k(t_k, u_k, z_k) < \infty$. Note that

$$\begin{aligned}\mathcal{E}_k(t_k,u_k,z_k) = \mathcal{E}_k(t,u_k,z_k) + \int_{\Omega_1\cup\Omega_2} \Big[&\mathbb{C}e(u_k){:}e(u_{\mathrm{D}}(t_k){-}u_{\mathrm{D}}(t)) \\ &+ \frac{1}{2}\mathbb{C}e(u_{\mathrm{D}}(t_k)){:}e(u_{\mathrm{D}}(t_k)) - \frac{1}{2}\mathbb{C}e(u_{\mathrm{D}}(t)){:}e(u_{\mathrm{D}}(t)) \\ &- (f(t_k){-}f(t)){\cdot}u_k - f(t_k){\cdot}u_{\mathrm{D}}(t_k) + f(t){\cdot}u_{\mathrm{D}}(t)\Big]\,\mathrm{d}x.\end{aligned}$$

Since $\|u_k\|_{H^1(\Omega_1\cup\Omega_2;\mathbb{R}^d)}$ is equibounded (by the same argument as in the Compactness part in the proof of Lemma 3.1), using also (2.4) and (2.8), we find that $\mathcal{E}_k(t_k,u_k,z_k) - \mathcal{E}_k(t,u_k,z_k)$ converges to zero as $k \to \infty$. Hence it is sufficient to apply the same arguments as in the Liminf part in the proof of Lemma 3.1.

Existence of an energetic solution (u,z) associated with $\mathcal{E}_\infty$ and $\mathcal{R}$ can now be stated and proved.

Theorem 3.3 ***(Existence of solutions to the original problem.)*** *Let the assumptions of Proposition 2.2 be satisfied. Assume in addition that* $q_0^k = (u_0^k, z_0^k)$ *satisfy* $q_0^k \overset{\mathcal{Q}}{\to} q_0$ *and* $\mathcal{E}_k(0,q_0^k) \to \mathcal{E}_\infty(0,q_0)$*. Then the energetic solutions* $q_k = (u_k,z_k)$ *for* $(\mathcal{E}_k, \mathcal{R}, q_0^k)$ *converge (in terms of subsequences) to an energetic solution* $q = (u,z)$ *associated with* $(\mathcal{E}_\infty, \mathcal{R}, q_0)$*, namely*

(i) $u_k(t) \rightharpoonup u(t)$ *in* $H^1(\Omega_1 \cup \Omega_2;\mathbb{R}^d)$ *for any* $t \in [0,T]$,

(ii) $z_k(t) \overset{*}{\rightharpoonup} z(t)$ *in* $L^\infty(\Gamma_{\mathrm{c}})$ *for any* $t \in [0,T]$,

(iii) $\mathcal{E}_k(t,u_k(t),z_k(t)) \to \mathcal{E}_\infty(t,u(t),z(t))$ *for any* $t \in [0,T]$,

(iv) $\partial_t\mathcal{E}_k(\cdot,u_k(\cdot),z_k(\cdot)) \to \partial_t\mathcal{E}_\infty(\cdot,u(\cdot),z(\cdot))$ *in* $L^1(0,T)$.

The proof consists of an application of Theorem 3.1 in [22]. The crucial assumption to be verified is the existence of a so-called joint recovery sequence. We recall the property that a joint recovery sequence $(\widehat{q}_{k_\ell})$ is required to satisfy.

$$\begin{aligned}&\forall\,(t_\ell, q_{k_\ell}) \text{ stable sequence with } (t_\ell,q_{k_\ell}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q), \quad \forall\,\widehat{q}\in\mathcal{Q}\ \ \exists\,\widehat{q}_{k_\ell} \overset{\mathcal{Q}}{\to} \widehat{q}: \\ &\limsup_{\ell\to\infty}\big(\mathcal{E}_{k_\ell}(t_\ell,\widehat{q}_{k_\ell}){+}\mathcal{R}(\widehat{q}_{k_\ell}{-}q_{k_\ell}){-}\mathcal{E}_{k_\ell}(t_\ell,q_{k_\ell})\big) \le \mathcal{E}_\infty(t,\widehat{q}){+}\mathcal{R}(\widehat{q}{-}q){-}\mathcal{E}_\infty(t,q),\end{aligned} \tag{3.9}$$

where, as in [22], we say that a sequence $(t_\ell, q_{k_\ell}) \in [0,T] \times \mathcal{Q}$ is a *stable sequence* if

$$q_{k_\ell} \text{ is stable for } (\mathcal{E}_{k_\ell},\mathcal{R}) \text{ at time } t_\ell \text{ and } \sup_{\ell\in\mathbb{N}} \mathcal{E}_{k_\ell}(t_\ell,q_{k_\ell}) < \infty. \tag{3.10}$$

Proof of Theorem 3.3: We first verify that the assumptions required in [22, Theorem 3.1] on the dissipation potential $\mathcal{R}$, on the energy functional $\mathcal{E}_\infty$ and on its power $\partial_t\mathcal{E}_\infty$ are easily satisfied in the present situation. Indeed, let us observe first that for stable sequences (t_ℓ, q_{k_ℓ}) and $(t_\ell, \widehat{q}_{k_\ell})$ with $(t_\ell,q_{k_\ell}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q)$, $(\widehat{t}_\ell, \widehat{q}_{k_\ell}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (\widehat{t},\widehat{q})$, we have

$$\mathcal{R}(\widehat{q}-q) \le \liminf_{\ell\to\infty} \mathcal{R}(\widehat{q}_{k_\ell} - q_{k_\ell}),$$

which corresponds to [22, condition (2.5)]. We then observe that, if $\mathcal{E}_k(t,u,z) < \infty$, then $\partial_t \mathcal{E}_k(t,u,z) = \partial_t \mathcal{V}(t,u)$ whose expression is given by (2.13). Hence, by using our assumptions (2.4) and (2.8) on the data, the following three properties (corresponding to [22, conditions (2.7)–(2.9)]) hold true:

Uniform control of $\partial_t \mathcal{E}_k$:
$$\exists\, c_0 \in \mathbb{R}\ \exists\, c_1 > 0\ \forall\, k \in \mathbb{N} \cup \{\infty\}\ \forall\, t \in [0,T]\ \forall\, q \in \mathcal{Q} \text{ with } \mathcal{E}_k(t,q) < \infty :$$
$$\mathcal{E}_k(\cdot,q) \in C^1([0,T]) \text{ and } \big|\partial_t \mathcal{E}_k(s,q)\big| \le c_1\big(c_0 + \mathcal{E}_k(s,q)\big) \quad \forall\, s \in [0,T].$$

Uniform time-continuity of the power $\partial_t \mathcal{E}_\infty$:
$$\forall\, \varepsilon > 0\ \forall\, E \in \mathbb{R}\ \exists\, \delta > 0 :$$
$$\mathcal{E}_\infty(t,q) \le E \text{ and } \big|t_1 - t_2\big| < \delta \Longrightarrow \big|\partial_t \mathcal{E}_\infty(t_1,q) - \partial_t \mathcal{E}_\infty(t_2,q)\big| < \varepsilon.$$

Conditioned continuous convergence of the power:
$$\forall \text{ stable sequence } (t_\ell, q_{k_\ell}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q) : \quad \partial_t \mathcal{E}_{k_\ell}(t_\ell, q_{k_\ell}) \to \partial_t \mathcal{E}_\infty(t,q).$$

Moreover, the lower Γ-limit for $\mathcal{E}_k$, which corresponds to [22, condition (2.10)], has been just proved in Lemma 3.1 and Corollary 3.2.

Let now $q_k = (u_k, z_k)$ be an energetic solution for $(\mathcal{E}_k, \mathcal{R}, q_0)$. We need to prove that $q_k(t) \overset{\mathcal{Q}}{\to} q(t)$ for all $t \in [0,T]$. First of all we derive suitable *a priori* estimates. By using $(0, z_k(t))$ as test for the stability condition (2.11) for $(\mathcal{E}_k, \mathcal{R})$ at time t, we obtain that $\mathcal{E}_k(t, u_k(t), z_k(t))$ is bounded uniformly with respect to t. By the assumptions (2.3), (2.4), and (2.8), we derive that $\|e(u_k(t))\|_{L^2(\Omega_1\cup\Omega_1;\mathbb{M}^{d\times d}_{\mathrm{sym}})}$ is bounded uniformly with respect to t. By Korn's inequality we obtain that there exists a positive constant C such that

$$\sup_{t\in[0,T]} \|u_k(t)\|_{H^1(\Omega_1\cup\Omega_2;\mathbb{R}^d)} \le C. \tag{3.11}$$

Moreover, since $0 \le z_k(t) \le 1$ on Γ_{c}, we have also that

$$\sup_{t\in[0,T]} \|z_k(t)\|_{L^\infty(\Gamma_{\mathrm{c}})} \le C, \tag{3.12}$$

for a possibly different constant C.

As the delamination process is unidirectional and thus monotone (and, in particular, also of bounded variation), we can apply the classical Helly's selection principle that gives a subsequence (not relabeled) and a limit function z non-increasing in time such that

$$z_k(t) \overset{*}{\rightharpoonup} z(t) \text{ in } L^\infty(\Gamma_{\mathrm{c}}) \qquad \forall t \in [0,T], \tag{3.13}$$

as claimed in the point (ii).

By (3.11), there exists a (possibly t-dependent) subsequence (see also, e.g., [8, 13]) u_k such that

$$u_k(t) \rightharpoonup u(t) \text{ in } H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d). \tag{3.14}$$

Yet, due to (2.1) and Korn's inequality, $u(t)$ is determined uniquely by $z(t)$ and thus there is no need of selecting a further subsequence. Hence, point (i) is proved.

In order to apply [22, Theorem 3.1] it remains to exhibit a joint recovery sequence as in (3.9). By [22, Proposition 2.2], in view of Lemma 3.1 and Corollary 3.2, we can prove the following condition, which implies (3.9):

$$\begin{aligned} &\forall \text{ stable sequence } (t_\ell, q_{k_\ell}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q),\ \forall\, \widehat{q} \in \mathcal{Q}\ \exists\, \widehat{q}_{k_\ell} \overset{\mathcal{Q}}{\to} \widehat{q} : \\ &\limsup_{\ell\to\infty} \big(\mathcal{E}_{k_\ell}(t_\ell, \widehat{q}_{k_\ell}) + \mathcal{R}(\widehat{q}_{k_\ell} - q_{k_\ell})\big) \le \mathcal{E}_\infty(t,\widehat{q}) + \mathcal{R}(\widehat{q} - q). \end{aligned} \tag{3.15}$$

So, let $\widehat{q} \in \mathcal{Q}$, let (t,q) be fixed and let (t_ℓ, q_{k_ℓ}) be a stable sequence with $(t_\ell, q_{k_\ell}) = (t_\ell, u_{k_\ell}, z_{k_\ell})$ converging to $(t,q) = (t,u,z)$. We need to construct a sequence $(t_\ell, \widehat{q}_{k_\ell}) =$

$(t_\ell, \widehat{u}_{k_\ell}, \widehat{z}_{k_\ell})$ satisfying (3.15). To this goal, we proceed similarly as in [21, Formula (5.41)]; cf. also [19, Lemma 6.1] where a closely related rate-independent delamination problem was considered. First of all we notice that the inequality (3.15) is nontrivial only when the right-hand side is finite, i.e., for $\widehat{q} = (\widehat{u}, \widehat{z})$ such that

$$0 \le \widehat{z} \le z, \tag{3.16a}$$

$$[\widehat{u}(x)]_{\Gamma_c} = 0 \text{ for a.e. } x \text{ such that } \widehat{z}(x) > 0. \tag{3.16b}$$

We claim that the recovery sequence $\widehat{q}_k$ in (3.15) can be taken as

$$\widehat{u}_{k_\ell} := \widehat{u}, \qquad \widehat{z}_{k_\ell} := \begin{cases} z_{k_\ell}\widehat{z}/z & \text{where } z > 0, \\ 0 & \text{where } z = 0. \end{cases} \tag{3.17}$$

Note that the convergence $z_{k_\ell} \overset{*}{\rightharpoonup} z$ in $L^\infty(\Gamma_c)$ assumed in (3.15) implies $\widehat{z}_{k_\ell} \overset{*}{\rightharpoonup} \widehat{z}$ in $L^\infty(\Gamma_c)$, as required in (3.15). Then it remains to verify the inequality for the limit superior in (3.15). We note that for the term $\mathcal{R}$ we have an equality in the limit, as

$$\lim_{\ell\to\infty} \mathcal{R}(\widehat{z}_{k_\ell} - z_{k_\ell}) = \lim_{\ell\to\infty} \int_{\Gamma_c} a_1(z_{k_\ell} - \widehat{z}_{k_\ell})\, \mathrm{d}\mathscr{H}^{d-1} = \int_{\Gamma_c} a_1(z - \widehat{z})\, \mathrm{d}\mathscr{H}^{d-1} = \mathcal{R}(\widehat{z} - z),$$

where we used that (3.16a) implies that $0 \le \widehat{z}_{k_\ell} \le z_{k_\ell}$ on Γ_c. Moreover, for $\mathcal{E}_{k_\ell}$ we have

$$\begin{aligned} \lim_{\ell\to\infty} \mathcal{E}_{k_\ell}(t_\ell, \widehat{q}_{k_\ell}) &= \lim_{\ell\to\infty} \int_{\Omega_1\cup\Omega_2} \frac{1}{2}\mathbb{C}e(\widehat{u}_{k_\ell} + u_{\mathrm{D}}(t_\ell)) : e(\widehat{u}_{k_\ell} + u_{\mathrm{D}}(t_\ell)) \\ &\qquad - f(t_\ell)\cdot(\widehat{u}_{k_\ell} + u_{\mathrm{D}}(t_\ell))\, \mathrm{d}x + \int_{\Gamma_c} (k_\ell [\widehat{u}_{k_\ell}]^2_{\Gamma_c} - a_0)\widehat{z}_{k_\ell}\, \mathrm{d}\mathscr{H}^{d-1} \\ &= \lim_{\ell\to\infty} \int_{\Omega_1\cup\Omega_2} \frac{1}{2}\mathbb{C}e(\widehat{u} + u_{\mathrm{D}}(t_\ell)) : e(\widehat{u} + u_{\mathrm{D}}(t_\ell)) \\ &\qquad - f(t_\ell)\cdot(\widehat{u} + u_{\mathrm{D}}(t_\ell))\, \mathrm{d}x + \int_{\Gamma_c} (k_\ell [\widehat{u}]^2_{\Gamma_c} - a_0)\widehat{z}_{k_\ell}\, \mathrm{d}\mathscr{H}^{d-1} \\ &= \mathcal{E}_\infty(t, \widehat{q}). \end{aligned} \tag{3.18}$$

Indeed, due to (3.17), it turns out that $k_\ell \widehat{z}_{k_\ell} [\widehat{u}]^2_{\Gamma_c} = 0$ on the set $\{\, x \in \Gamma_c \,|\, z(x) = 0 \,\}$. Actually, due to (3.16b), the same equality holds true also on the (complementary) set $\{\, x \in \Gamma_c \,|\, z(x) > 0 \,\}$. Therefore $\widehat{q}_{k_\ell} = (\widehat{u}_{k_\ell}, \widehat{z}_{k_\ell})$ is indeed the sought joint recovery sequence.

Altogether, we can apply [22, Theorem 3.1], showing that (u, z) is an energetic solution to the original problem and also that points (iii)–(iv) hold true. ∎

4 Concluding remarks

Remark 4.1 (Classical formulation of the regularized problem.) The energetic formulation is related with the doubly-nonlinear evolution inclusions of the type $\partial\mathcal{R}(\frac{\partial q}{\partial t}) + \partial_q \mathcal{E}_k(t, q) \ni 0$, for $k \in \mathbb{N} \cup \{\infty\}$, cf. [20,24]. In the case of the regularized problem governed by $\mathcal{E}_k$ and $\mathcal{R}$ it is even possible to rewrite the differential inclusions in a classical formulation in terms of complementarity problems, like in [18]. More precisely, for given $k \in \mathbb{N}$, an energetic solution (u, z) associated with $\mathcal{E}_k$ and $\mathcal{R}$

satisfies the following equations, where we set $\widetilde{u} = u + u_{\mathrm{D}}$:

$$
\begin{aligned}
&-\operatorname{div}\big(\mathbb{C}e(\widetilde{u}(t))\big) = f(t) && \text{on } \Omega_1 \cup \Omega_2, && \text{(4.1a)}\\
&\widetilde{u}(t) = w_{\mathrm{D}}(t) && \text{on } \Gamma_{\mathrm{D}}, && \text{(4.1b)}\\
&T(\widetilde{u}(t)) = 0 && \text{on } \Gamma_{\mathrm{N}}, && \text{(4.1c)}\\
&\left.\begin{aligned}
&\big[\mathbb{C}e(\widetilde{u}(t))\big]_{\Gamma_{\mathrm{c}}}\nu = 0\\
&2kz(t)\widetilde{u}_{\mathrm{t}}(t) - T_{\mathrm{t}}(\widetilde{u}(t)) = 0\\
&\widetilde{u}_{\mathrm{n}}(t) \geq 0\\
&2kz(t)\widetilde{u}_{\mathrm{n}}(t) - T_{\mathrm{n}}(\widetilde{u}(t)) \geq 0\\
&\big(2kz(t)\widetilde{u}_{\mathrm{n}}(t) - T_{\mathrm{n}}(\widetilde{u}(t))\big)\widetilde{u}_{\mathrm{n}}(t) = 0\\
&\dot{z}(t) \leq 0\\
&k[\widetilde{u}(t)]^2_{\Gamma_{\mathrm{c}}} - r(z(t)) \leq a_0 + a_1\\
&\dot{z}(t)(k[\widetilde{u}(t)]^2_{\Gamma_{\mathrm{c}}} - r(z(t)) - a_0 - a_1) = 0
\end{aligned}\right\} && \text{on } \Gamma_{\mathrm{c}}. && \text{(4.1d)}
\end{aligned}
$$

In the previous equations, $T(\widetilde{u}) = \mathbb{C}e(\widetilde{u})|_{\Gamma}\nu$ denotes the traction stress on $\Gamma = \Gamma_{\mathrm{c}}$ or $\Gamma = \Gamma_{\mathrm{N}}$. Moreover, its normal and tangential components on Γ_{c} are denoted with $T_{\mathrm{n}}(\widetilde{u}) = (T(\widetilde{u})\cdot\nu)\nu$ and $T_{\mathrm{t}}(\widetilde{u}) = T(\widetilde{u}) - (T(\widetilde{u})\cdot\nu)\nu$, respectively, so that we have the decomposition $T(\widetilde{u}) = T_{\mathrm{n}}(\widetilde{u}) + T_{\mathrm{t}}(\widetilde{u})$. Similarly we decompose the jump $[\widetilde{u}]_{\Gamma_{\mathrm{c}}}$ into normal and tangential components, as $[\widetilde{u}]_{\Gamma_{\mathrm{c}}} = \widetilde{u}_{\mathrm{n}} + \widetilde{u}_{\mathrm{t}}$. We remark that, since by our choice ν turns out to be the inner unit normal on Ω_1, in (4.1d) there is a minus sign in front of $T_{\mathrm{t}}(\widetilde{u})$ and $T_{\mathrm{n}}(\widetilde{u})$. Note also that, due to $[\mathbb{C}e(\widetilde{u})]_{\Gamma_{\mathrm{c}}}\nu = 0$ in (4.1d), the traction stress $T(\widetilde{u})$ is well defined on Γ_{c} by the formula $T(\widetilde{u}) = \mathbb{C}e(\widetilde{u})|_{\Gamma}\nu$, no matter from which side of Γ_{c} the limit is taken. Moreover, $r(z) \in N_{[0,1]}(z)$ with $N_{[0,1]}$ denoting the normal cone to $[0,1]$. On the other hand, a classical formulation of the original problem governed by $\mathcal{E}_\infty$ and $\mathcal{R}$ does not seem obvious. This might be due to loss of a linear-space structure of this problem.

Remark 4.2 (Thermodynamical context.) The classical formulation (4.1) reveals that, from a purely mechanical viewpoint, only the sum $a_0 + a_1 =: a$ plays a role, and that such a has to be understood as an *activation energy* per unit area needed to trigger the delamination process. This effect is due to the irreversibility of the delamination process. Indeed, the stored energy $a_0(1-z)$ cannot be gained back and is thus forever dissipated as well as the contribution $a_1(1-z)$ in the dissipation potential. However, if the full thermodynamically consistent system of the mechanical equations/inequalities like (4.1) and of a heat equation is considered, only the part a_1 contributes to the entropy and thus to the heat production, while the part a_0 is merely nondissipative in the sense that this part of total energy a spent by delamination is irreversibly stored into the delaminated surface itself. This suggested the splitting of a into $a_0 + a_1$ adopted in this paper. We underline that the case $a_1 = 0$ has been considered in [14], while the opposite case $a_0 = 0$ has been adopted in [18].

Remark 4.3 (Materials with nonlinear response.) Instead of a quadratic stored energy as in (2.2), corresponding to a linearly elastic material, one can consider a nonquadratic stored energy $W : \mathbb{M}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ of class C^1 satisfying, for some $1 < p < \infty$ and some constants $c_i > 0$, the following estimate

$$
\forall A \in \mathbb{M}^{d\times d}_{\mathrm{sym}} : \qquad c_1|A|^p - c_2 \leq W(A) \leq c_3(1 + |A|^p). \tag{4.2}
$$

With this choice, the corresponding energy functionals $\mathcal{E}_\infty$ and $\mathcal{E}_k$ are defined as in (2.10a) and (2.14), where the stored bulk energy in (2.10b) is replaced with

$$
\mathcal{V}(t,u) := \int_{\Omega_1\cup\Omega_2} W(e(u+u_{\mathrm{D}}(t))) - f(t)\cdot(u+u_{\mathrm{D}}(t))\,\mathrm{d}x. \tag{4.3}
$$

The mathematical arguments used above are preserved essentially as far as the weak lower semicontinuity of $\mathcal{V}(t,\cdot)$ is guaranteed. This happens, e.g., if W is quasiconvex. Then, from (4.2), one can derive an estimate for the derivative $\mathrm{D}W$ of the energy density, namely

$$\forall A \in \mathbb{M}^{d\times d}_{\mathrm{sym}}: \qquad |\mathrm{D}W(A)| \le c_4(1+|A|^{p-1})$$

for some constant $c_4 > 0$. Under these assumptions, for the stored bulk energy $\mathcal{V}$ given in (4.3), the existence of an energetic solution (u_k, z_k) associated with $\mathcal{E}_k$ and $\mathcal{R}$ can be derived, as the assumptions of the general Theorem 4.5 of [19] are satisfied. Moreover, it is easy to verify that $\mathcal{E}_k$ Γ-converges to $\mathcal{E}_\infty$. The results of Section 3 can be therefore adapted to this situation, and the existence of an energetic solution (u, z) associated with $\mathcal{E}_\infty$ and $\mathcal{R}$ can be deduced, as an application of [22, Theorem 3.1]. Indeed, the existence of the joint recovery sequence was obtained by the lower semicontinuity of the elastic energy, which is guaranteed also in this case. As pointed out already by L. Tartar, the weak lower semicontinuity of $\mathcal{V}(t,\cdot)$ follows, in fact, from a weaker assumption on W, the so-called $\mathcal{A}$-quasiconvexity (cf. [6,11,12]) for $\mathcal{A}$ being an operator whose kernel consists of all symmetric rotation-free fields.

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